

Direct and inverse images for fractional stochastic tangent sets and applications

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Abstract

In this paper, we study direct and inverse images for fractional stochastic tangent sets and we establish the deterministic necessary and sufficient conditions that guarantee that the solution of a given stochastic differential equation driven by the fractional Brownian motion evolves in some particular sets K . As a consequence, a comparison theorem is obtain.

Keywords: Stochastic Viability, Stochastic Differential Equations, Stochastic Tangent Sets, Fractional Brownian Motion.

1 Introduction

A general result on the existence and uniqueness of the solution for multidimensional, time dependent, stochastic differential equations (SDEs) driven by a fractional Brownian motion (fBm) with Hurst parameter $H > \frac{1}{2}$ has been given by Nualart and Răşcanu in [14] using a techniques of the classical fractional calculus.

The notion of viable trajectories, used in the theory of deterministic and stochastic differential equations, refers to those trajectories which remain at any time in a fixed subset of the state space. The viability is to find necessary and sufficient conditions such that a fixed subset is viable for the differential equation. In the theory of viable solutions the concept of the tangent sets and contingent sets play a fundamental role. In fact, the pioneering theorem, proved in 1942 by Nagumo, gives a criterion of the viability in terms of contingent sets. Namely, the Nagumo theorem states that if f is a bounded, continuous map from a closed subset K of \mathbb{R}^m to \mathbb{R}^m , then a necessary and sufficient condition such that K is viable for the differential equation

$$x'(t) = f(x(t)), \quad x(0) = x_0 \in K.$$

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is that

$$\langle f(x), p \rangle \leq 0, \quad \forall x \in K \text{ and } \forall p \text{ a normal vector at } K \text{ in } x.$$

Various generalizations of the Nagumo theorem provide viability conditions in terms of contingent cones (see for instance [1] Th. 1, p. 191). Viability and invariance with respect to Itô equations have been investigated first by J.-P. Aubin and G. Da Prato in [3]. Criteria for the viability and invariance of closed and convex subset of \mathbb{R}^m , given in [3], are expressed in terms of stochastic contingent sets. Their results were generalized to arbitrary subsets (which can also be time-dependent and random) in [12].

Another approach has been developed by Buckdahn, Peng, Quincampoix, Rainer and Răşcanu in [6], [7], [8], [9]. The main point of their work consist in proving that the viability property for SDE and also for backward SDE holds true if and only if the square of the distance to the constraint sets is a viscosity supersolution(subsolution) of the Hamilton-Jacobi-Bellman equation associated.

With respect to the SDE driven by fBm, I. Ciotir and A. Răşcanu proved a type of Nagumo Theorem on viability properties of close bounded subsets with respect to a stochastic differential equation driven by fractional Brownian motion in [10].

Conditions expressed by stochastic contingent sets which are given in [10] are general but unfortunately not easy to check and the aim of the present paper is to give checkable conditions for general stochastic differential equation driven by the fractional Brownian motion and some particular sets K .

Studying from [10], we find the deterministic necessary and sufficient conditions that guarantee that the solution of a stochastic differential equation driven by the fractional Brownian motion B^H with Hurst parameter $\frac{1}{2} < H < 1$ (in short: f-SDE), \mathbb{P} -a.s. $\omega \in \Omega$

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}) dB_r^H, \quad s \in [t, T],$$

$(t, x) \in [0, T] \times \mathbb{R}^d$ involves in some particular sets K i.e. under which it holds that for all $t \in [0, T]$ and for all $x \in K$:

$$X_s^{t,x} \in K \quad \text{a.s. } \omega \in \Omega, \quad \forall s \in [t, T].$$

Here

- $B^H = \{B_t^H, t \geq 0\}$ is a fractional Brownian motion with Hurst parameter $\frac{1}{2} < H < 1$, and the integral with respect to B^H is a pathwise Riemann-Stieltjes integral;
- $b(t, x) : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma(t, x) : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ are continuous functions.

The characterization of viability of K is obtained through the study of the direct and inverse images for fractional stochastic tangent sets. This idea comes from [3]. In fact we extend the direct and inverse images of stochastic tangent sets to the fractional form and using our main theorem 3.2, we character the viability of some particular sets K with the conditions on b and σ and we also obtain a comparison theorem.

We now explain how the paper is organized. In the second section, we recall some classical definitions and the assumptions on the coefficients supposed to hold. we also recall the main result in [10], which we will use later. In section 3 we state our main result and some applications are given. The section 4 is devoted to the proof of the main result and section 5 is for the proof of a general comparison theorem.

2 Preliminaries

Consider the equation on \mathbb{R}^d

$$X_s = X_0 + \int_0^s b(r, X_r) dr + \int_0^s \sigma(r, X_r) dB_r^H, \quad s \in [0, T], \quad (2.1)$$

- $B^H = \{B_t^H, t \geq 0\}$ is a fractional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$; with Hurst parameter $\frac{1}{2} < H < 1$, and the integral with respect to B^H is a pathwise Riemann-Stieltjes integral;
- X_0 is a d - dimensional random variable.
- $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ are continuous functions.

Remark that the fractional Brownian motion has the following property:

For every $0 < \varepsilon < H$ and $T > 0$ there exists a positive random variable $\eta_{\varepsilon, T}$ such that $\mathbb{E}(|\eta_{\varepsilon, T}|^p) < \infty$, for all $p \in [1, \infty)$ and for all $s, t \in [0, T]$

$$|B^H(t) - B^H(s)| \leq \eta_{\varepsilon, T} |t - s|^{H-\varepsilon} \quad a.s.$$

And from [5] proposition 1.7.1 (see also in [11]), we have for every $t_0 \in [0, +\infty)$,

$$\mathbb{P} \left\{ \limsup_{t \rightarrow t_0, t \geq t_0} \left| \frac{B^H(t) - B^H(t_0)}{t - t_0} \right| = +\infty \right\} = 1 \quad (2.2)$$

Using the same method we can easily proof that

$$\mathbb{P} \left\{ \limsup_{t \rightarrow t_0, t \geq t_0} \frac{B^H(t) - B^H(t_0)}{t - t_0} = +\infty \right\} = \mathbb{P} \left\{ \liminf_{t \rightarrow t_0, t \geq t_0} \frac{B^H(t) - B^H(t_0)}{t - t_0} = -\infty \right\} = \frac{1}{2} \quad . \quad (2.3)$$

2.1 Assumptions and Notations

For the function and coefficients appearing in the equation (2.1), we make the following standard assumptions which we will use throughout the paper:

(H₁) $\sigma(t, x)$ is differentiable in $x \in \mathbb{R}^d$, and there exist some constants $\beta, \delta, 0 < \beta, \delta \leq 1$, and for every $R > 0$ there exists $M_R > 0$ such that the following properties hold for all $t \in [0, T]$,

$$(H_\sigma) : \begin{cases} i) & |\sigma(t, x) - \sigma(s, y)| \leq M_0 (|t - s|^\beta + |x - y|), \quad \forall x, y \in \mathbb{R}^d, \\ ii) & |\nabla_x \sigma(t, y) - \nabla_x \sigma(s, z)| \leq M_R (|t - s|^\beta + |y - z|^\delta), \quad \forall |y|, |z| \leq R, \end{cases}$$

where $\nabla_x \sigma(t, x) = (\nabla_x \sigma^i(t, x))_{i=1, \dots, d}$ and

$$|\nabla_x \sigma(t, x)|^2 = \sum_{l=1}^d \sum_{i=1}^d |\partial_{x_l} \sigma^i(t, x)|^2$$

Remark that for all $x \in \mathbb{R}^d$

$$|\sigma(t, x)| \leq |\sigma(0, 0)| + M_0 (|t|^\beta + |x|) \leq M_{0, T} (1 + |x|)$$

where $M_{0, T} = |\sigma(0, 0)| + M_0 + M_0 T$.

Let

$$\alpha_0 = \min \left\{ \frac{1}{2}, \beta, \frac{\delta}{1 + \delta} \right\}.$$

(H₂) There exist $\mu \in (1 - \alpha_0, 1]$ and for every $R \geq 0$ there exists $L_R > 0$ such that the following properties hold for all $t \in [0, T]$,

$$(H_b) : \begin{cases} i) & |b(r, x) - b(s, y)| \leq L_R (|r - s|^\mu + |x - y|), \quad \forall |x|, |y| \leq R, \\ ii) & |b(t, x)| \leq L_0(1 + |x|), \quad \forall x \in \mathbb{R}^d. \end{cases}$$

Finally, we introduce some notations which will be used later.

Let $d, k \in \mathbb{N}^*$. Given a matrix $A = (a^{i,j})_{d \times k}$ and a vector $y = (y^i)_{d \times 1}$, we denote $|A|^2 = \sum_{i,j} |a^{i,j}|^2$ and $|y| = \sum_i |y^i|^2$.

Let $t \in [0, T]$ be fixed. Denote

- $W^{\alpha, \infty}(t, T; \mathbb{R}^d)$, $0 < \alpha < 1$, the space of continuous functions $f : [t, T] \rightarrow \mathbb{R}^d$ such that

$$\|f\|_{\alpha, \infty; [t, T]} := \sup_{s \in [t, T]} \left(|f(s)| + \int_t^s \frac{|f(s) - f(r)|}{(s - r)^{\alpha+1}} dr \right) < \infty.$$

An equivalent norm can be defined by

$$\|f\|_{\alpha, \lambda; [t, T]} := \sup_{s \in [t, T]} e^{-\lambda s} \left(|f(s)| + \int_t^s \frac{|f(s) - f(r)|}{(s - r)^{\alpha+1}} dr \right) \quad \forall \lambda \geq 0.$$

- $\tilde{W}^{1-\alpha, \infty}(t, T; \mathbb{R}^d)$, $0 < \alpha < \frac{1}{2}$, the space of continuous functions $g : [t, T] \rightarrow \mathbb{R}^d$ such that

$$\|g\|_{\tilde{W}^{1-\alpha, \infty}(t, T; \mathbb{R}^d)} := |g(t)| + \sup_{t < r < s < T} \left(\frac{|g(s) - g(r)|}{(s - r)^{1-\alpha}} + \int_r^s \frac{|g(y) - g(r)|}{(y - r)^{2-\alpha}} dy \right) < \infty.$$

- $C^\mu([t, T]; \mathbb{R}^d)$, $0 < \mu < 1$, the space of μ -Hölder continuous functions $f : [t, T] \rightarrow \mathbb{R}^d$, equipped with the norm

$$\|f\|_{\mu; [t, T]} := \|f\|_{\infty; [t, T]} + \sup_{t \leq r < s \leq T} \frac{|f(s) - f(r)|}{(s - r)^\mu} < \infty$$

where $\|f\|_{\infty; [t, T]} := \sup_{s \in [t, T]} |f(s)|$. We have, for all $0 < \varepsilon < \alpha$

$$C^{\alpha+\varepsilon}([t, T]; \mathbb{R}^d) \subset W^{\alpha, \infty}(t, T; \mathbb{R}^d)$$

- $W^{\alpha, 1}(t, T; \mathbb{R}^d)$ the space of measurable functions f on $[t, T]$ such that

$$\|f\|_{\alpha, 1; [t, T]} := \int_t^T \left[\frac{|f(s)|}{(s - t)^\alpha} + \int_t^s \frac{|f(s) - f(y)|}{(s - y)^{\alpha+1}} dy \right] ds < \infty.$$

Clearly

$$W^{\alpha, \infty}(t, T; \mathbb{R}^d) \subset W^{\alpha, 1}(t, T; \mathbb{R}^d).$$

2.2 Generalized Stieltjes integral

Denoting

$$\Lambda_\alpha(g; [t, T]) := \frac{1}{\Gamma(1-\alpha)} \sup_{t < r < s < T} |(D_{s-}^{1-\alpha} g_{s-})(r)|.$$

where

$$\Gamma(\alpha) = \int_0^\infty s^{\alpha-1} e^{-s} ds$$

is the Gamma function and

$$(D_{s-}^{1-\alpha} g_{s-})(r) = \frac{e^{i\pi(1-\alpha)}}{\Gamma(\alpha)} \left(\frac{g(s) - g(r)}{(s-r)^{1-\alpha}} + (1-\alpha) \int_r^s \frac{g(r) - g(y)}{(y-r)^{2-\alpha}} dy \right) 1_{(t,s)}(r).$$

we have

$$\Lambda_\alpha(g; [t, T]) \leq \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \|g\|_{\tilde{W}^{1-\alpha,\infty}(t,T;\mathbb{R}^d)}$$

Note that

$$\Lambda_\alpha(g; [t, T]) \leq \Lambda_\alpha(g; [0, T]) \left(:= \Lambda_\alpha(g) \right).$$

We also introduce the notation

$$(D_{t+}^\alpha f)(r) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(r)}{(r-t)^\alpha} + \alpha \int_t^r \frac{f(r) - f(y)}{(r-y)^{\alpha+1}} dy \right) 1_{(t,T)}(r).$$

Definition 2.1 Let $0 < \alpha < \frac{1}{2}$. If $f \in W^{\alpha,1}(t, T; \mathbb{R}^{d \times k})$ and $g \in \tilde{W}^{1-\alpha,\infty}(t, T; \mathbb{R}^k)$, then defining

$$\int_t^s f(r) dg(r) := (-1)^\alpha \int_t^s (D_{t+}^\alpha f)(r) (D_{s-}^{1-\alpha} g_{s-})(r) dr.$$

the integral $\int_t^s f dg$ exists for all $s \in [t, T]$ and

$$\begin{aligned} \left| \int_t^T f(r) dg(r) \right| &\leq \sup_{t \leq r < s \leq T} |(D_{s-}^{1-\alpha} g_{s-})(r)| \int_t^T |(D_{t+}^\alpha f)(s)| ds \\ &\leq \Lambda_\alpha(g; [t, T]) \|f\|_{\alpha,1;[t,T]}. \end{aligned}$$

It is known that when $H \in (\frac{1}{2}, 1)$ and $1 - H < \alpha < \frac{1}{2}$, then the random variable

$$G = \Lambda_\alpha(B^H) = \frac{1}{\Gamma(1-\alpha)} \sup_{t < s < r < T} |(D_{r-}^{1-\alpha} B_{r-})(s)|$$

has moments of all order. As a consequence, if $u = \{u_t, t \in [0, T]\}$ is a stochastic process whose trajectories belong to the space $W^{\alpha,1}(t, T; \mathbb{R}^d)$, with $1 - H < \alpha < \frac{1}{2}$, the pathwise integral

$\int_0^T u_s dB_s^H$ exists in the sense of Definition 2.1 and we have the estimate

$$\left| \int_0^T u_s dB_s^H \right| \leq G \|u\|_{\alpha,1}.$$

This is the reason why in the SDE (2.1) the integral with respect to B^H is a pathwise Riemann-Stieltjes integral.

D. Nualart and A. Răşcanu have proved in [14] that under the assumptions **(H₁)** and **(H₂)**, with $\beta > 1 - H$ and $\delta > \frac{1}{H} - 1$ the SDE

$$X_s^{t,\xi} = \xi + \int_t^s b(r, X_r^{t,\xi}) dr + \int_t^s \sigma \left(r, X_r^{t,\xi} \right) dB_r^H, \quad s \in [t, T],$$

has a unique solution $X^{t,\xi} \in L^0(\Omega, \mathcal{F}, \mathbb{P}; W^{\alpha,\infty}(t, T; \mathbb{R}^d))$, for all $\alpha \in (1 - H, \alpha_0)$. Moreover, for \mathbb{P} -almost all $\omega \in \Omega$, $X(\omega, \cdot) \in C^{1-\alpha}(0, T; \mathbb{R}^d)$.

2.3 Fractional Viability

In this subsection we recall the notion of the viability property for SDE driven by fractional Brownian motion. On the other hand we will present the main result of [10] which is very useful for our results.

Consider the stochastic differential equation driven by fractional Brownian motion B^H with Hurst parameter $\frac{1}{2} < H < 1$,

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}) dB_r^H, \quad s \in [t, T]. \quad (2.4)$$

Definition 2.2 Let $\mathcal{K} = \{K(t) : t \in [0, T]\}$ be a family of subsets of \mathbb{R}^d . We say that \mathcal{K} is viable (weak invariant) for the equation (2.4) if, starting at any time $t \in [0, T]$ and from any point $x \in K(t)$, there exists at least one of its solutions $\{X_s^{t,x} : s \in [t, T]\}$ which satisfies

$$X_s^{t,x} \in K(s) \quad \text{for all } s \in [t, T].$$

Definition 2.3 The family \mathcal{K} is invariant (strong invariant) for the equation (2.4) if, for any $t \in [0, T]$ and for any starting point $x \in K(t)$, all solutions $\{X_s^{t,x} : s \in [t, T]\}$ of the fractional stochastic differential equation (2.4) have the property

$$X_s^{t,x} \in K(s) \quad \text{for all } s \in [t, T].$$

Remark that, in the case when the equation has a unique solution (which is the case for equation (2.4) under the assumptions (\mathbf{H}_1) and (\mathbf{H}_2)), viability is equivalent to invariance.

Assuming that the mappings b and σ from the equation (2.4) satisfy (\mathbf{H}_1) and (\mathbf{H}_2) .

Definition 2.4 Let $t \in [0, T]$ and $x \in K(t)$. Let $\frac{1}{2} < 1 - \alpha < H$.

The $(1 - \alpha)$ -fractional B^H -contingent set to $K(t)$ in x is the set of the pairs (u, v) , such that there exist random variable $\bar{h} = \bar{h}^{t,x} > 0$ and a stochastic process $Q = Q^{t,x} : \Omega \times [t, t + \bar{h}] \rightarrow \mathbb{R}^d$, and for every $R > 0$ with $|x| \leq R$ there exist two random variables $H_R, \tilde{H}_R > 0$ independent of (t, \bar{h}) and a constant $\gamma = \gamma_R(\alpha, \beta) \in (0, 1)$ such that for all $s, \tau \in [t, t + \bar{h}]$, \mathbb{P} -a.s.

$$|Q(s) - Q(\tau)| \leq H_R |s - \tau|^{1-\alpha}, \quad |Q(s)| \leq \tilde{H}_R |s - t|^{1+\gamma}$$

and

$$x + (s - t)u + v [B_s^H - B_t^H] + Q(s) \in K(s),$$

where the constants H_R, \tilde{H}_R depend only on $R, L_R, M_{0,T}, M_0, L_0, T, \alpha, \beta, \Lambda_\alpha(B^H)$.

Definition 2.5 Let $t \in [0, T]$ and $x \in K(t)$. Let $\frac{1}{2} < 1 - \alpha < H$.

The $(1 - \alpha)$ -fractional B^H -tangent set to $K(t)$ in x , denoted by $S_{K(t)}(t, x)$, is the set of the pairs (u, v) , such that there exist random variable $\bar{h} = \bar{h}^{t,x} > 0$ and two stochastic process

$$U = U^{t,x} : [t, t + \bar{h}] \rightarrow \mathbb{R}^d, \quad U(t) = 0$$

$$V = V^{t,x} : [t, t + \bar{h}] \rightarrow \mathbb{R}^d, \quad V(t) = 0$$

and for every $R > 0$ with $|x| \leq R$ there exist two random variables $D_R, \tilde{D}_R > 0$ independent of (t, \bar{h}) such that for all $s, \tau \in [t, t + \bar{h}]$, \mathbb{P} -a.s.

$$|U(s) - U(\tau)| \leq D_R |s - \tau|^{1-\alpha}, \quad |V(s) - V(\tau)| \leq \tilde{D}_R |s - \tau|^{\min\{\beta, 1-\alpha\}}$$

and

$$x + \int_t^s (u + U(r)) dr + \int_t^s (v + V(r)) dB_r^H \in K(s),$$

where the constants D_R, \tilde{D}_R depend only on $R, L_R, M_{0,T}, M_0, L_0, T, \alpha, \beta, \Lambda_\alpha(B^H)$.

Remark.

- From [10], we can always assume that $0 < \bar{h} \leq 1$.
- The definition of $S_{\varphi(K(t))}(t, \varphi(x))$ is the same to $S_{K(t)}(t, x)$, only changes the condition

$$x + \int_t^s (u + U(r))dr + \int_t^s (v + V(r))dB_r^H \in K(s),$$

to

$$\varphi(x) + \int_t^s (u + U(r))dr + \int_t^s (v + V(r))dB_r^H \in \varphi(K(s)).$$

Now we recall the main result of [10] concerning the stochastic viability.

Theorem 2.6 *Let $\mathcal{K} = \{K(t) : t \in [0, T]\}$, $K(t) = \overline{K(t)} \subset \mathbb{R}^d$. Assume that (\mathbf{H}_1) and (\mathbf{H}_2) are satisfied with $\frac{1}{2} < H < 1$, $1 - H < \beta$, $\delta > \frac{1-H}{H}$. Let $1 - H < \alpha < \alpha_0$. Then the following assertions are equivalent:*

- (I) \mathcal{K} is viable for the fractional SDE, i.e. for all $t \in [0, T]$ and for all $x \in K(t)$ there exists a solution $X^{t,x}(\omega, \cdot) \in C^{1-\alpha}([t, T]; \mathbb{R}^d)$ of the equation

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dB_r^H, \quad s \in [t, T], \quad a.s. \omega \in \Omega,$$

and

$$X_s^{t,x} \in K(s), \quad \forall s \in [t, T].$$

- (II) For all $t \in [0, T]$ and all $x \in K(t)$, $(b(t, x), \sigma(t, x))$ is $(1 - \alpha)$ -fractional B^H -contingent to $K(t)$ in x .
- (III) For all $t \in [0, T]$ and all $x \in K(t)$, $(b(t, x), \sigma(t, x))$ is $(1 - \alpha)$ -fractional B^H -tangent to $K(t)$ in x .

Remark. The assertion (III) is given only for the deterministic case in [10]. In fact we can obtain the stochastic case from the deterministic one in the same manner as that (II) is obtained.

Under the same assumptions in Theorem 2.6, it follows:

Corollary 2.7 *If K is independent of t , the following assertions are equivalent:*

- (j) K is viable for the fractional SDE (2.4).
- (jj) For all $t \in [0, T]$ and all $x \in \partial K$, $(b(t, x), \sigma(t, x))$ is $(1 - \alpha)$ -fractional B^H -contingent to K in x .
- (jjj) For all $t \in [0, T]$ and all $x \in \partial K$, $(b(t, x), \sigma(t, x))$ is $(1 - \alpha)$ -fractional B^H -tangent to K in x .

Proof. When K is independent of t , just using Theorem 2.6, it's obvious that $(j) \Rightarrow (jj) \Rightarrow (jjj)$. Now we only need prove $(jjj) \Rightarrow (j)$, In fact we will prove $(jjj) \Rightarrow (III)$, and then we will get our result.

Let $t \in [0, T]$ and $\forall x \in K \setminus \partial K$, Since $X^{t,x}$ is continuous, then there exists a random variable \bar{h} , such that for all $s \in [t, t + \bar{h}]$,

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dB_r^H \in K.$$

we have for all $s \in [t, t + \bar{h}]$,

$$X_s^{t,x} = x + \int_t^s [b(t, x) + U(r)]dr + \int_t^s [\sigma(t, x) + V(r)]dB_r^H \in K.$$

where

$$U(r) = b(r, X_r^{t,x}) - b(t, x), \quad V(r) = \sigma(r, X_r^{t,x}) - \sigma(t, x)$$

clearly that $(b(t, x), \sigma(t, x))$ is $(1 - \alpha)$ -fractional B^H -tangent to K in x . Together with (jjj), we have that for all $t \in [0, T]$ and all $x \in K$, $(b(t, x), \sigma(t, x))$ is $(1 - \alpha)$ -fractional B^H -tangent to K in x . This is just (III) for the case that K is independent of t . □

3 Results and Applications

The next two theorems are our main theorems, firstly we extend *Stochastic Tangent Sets to Direct Images* which is introduced by J.P.Aubin, and G.Da Prato [3] (1990) to the fBM framework.

Theorem 3.1 *Assume that (\mathbf{H}_1) and (\mathbf{H}_2) are satisfied. Let $K(t) = \overline{K(t)} \subset \mathbb{R}^d, t \in [0, T]$ and $S_{K(t)}(t, x)$ the $(1 - \alpha)$ -fractional B^H -tangent set to K in x . Let φ be a C^2 map from \mathbb{R}^d to \mathbb{R}^m with a bounded second derivative. If*

$$(b(t, x), \sigma(t, x)) \in S_{K(t)}(t, x)$$

then

$$(\varphi'(x)b(t, x), \varphi'(x)\sigma(t, x)) \in S_{\varphi(K(t))}(t, \varphi(x)).$$

Also we can prove the *Stochastic Tangent Sets to Inverse Images* in the fBM form.

We introduce a space \mathcal{H} of the functions $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^m$ of class C^2 , with a bounded and Lipschitz continuous second derivative and there exist $a_\varphi < b_\varphi$ and some constants $M > 0$, $L > 0$ such that for all $a_\varphi \leq |x| \leq b_\varphi$, the matrix $\varphi'(x)$ has a right inverse denoted by $\varphi'(x)^+$ satisfying

$$(1) \quad |[\varphi'(x)^+]'| \leq M,$$

$$(2) \quad |[\varphi'(x)^+]' - [\varphi'(y)^+]'| \leq L|x - y|.$$

Theorem 3.2 *Assume that (\mathbf{H}_1) and (\mathbf{H}_2) are satisfied. Let $K(t) = \overline{K(t)} \subset \mathbb{R}^d, t \in [0, T]$ and $\varphi \in \mathcal{H}$, then for every $\varepsilon > 0$ and $a_\varphi + \varepsilon \leq |x| \leq b_\varphi - \varepsilon$, then*

$$(b(t, x), \sigma(t, x)) \in S_{\varphi^{-1}(\varphi(K(t)))}(t, x)$$

if and only if

$$(\varphi'(x)b(t, x), \varphi'(x)\sigma(t, x)) \in S_{\varphi(K(t))}(t, \varphi(x)).$$

Using Theorem 3.2, we can get the deterministic sufficient and necessary conditions for viability when K takes some particular forms. Firstly we give some Lemmas.

Lemma 3.3 *Let K be the unit sphere, then for all $x \in K$, $(b(t, x), \sigma(t, x)) \in S_K(t, x)$ if and only if*

$$\langle x, b(t, x) \rangle = 0, \quad \langle x, \sigma(t, x) \rangle = 0.$$

Lemma 3.4 Let $K = \{x \in \mathbb{R}^d; r \leq |x| \leq R\}$ then for all x , such that $|x| = R$, $(b(t, x), \sigma(t, x)) \in S_K(t, x)$ if and only if

$$\langle x, b(t, x) \rangle \leq 0, \quad \langle x, \sigma(t, x) \rangle = 0$$

and for all x , such that $|x| = r$, $(b(t, x), \sigma(t, x)) \in S_K(t, x)$ if and only if

$$\langle x, b(t, x) \rangle \geq 0, \quad \langle x, \sigma(t, x) \rangle = 0.$$

Lemma 3.5 Let K be the unit ball, then for all x , such that $|x| = 1$, $(b(t, x), \sigma(t, x)) \in S_K(t, x)$ if and only if

$$\langle x, b(t, x) \rangle \leq 0, \quad \langle x, \sigma(t, x) \rangle = 0.$$

Just as Corollary 2.7 said, considering that if we want to get the conditions for the viability of K , we only need to think about the starting point $x \in \partial K$. Then together with Lemma 3.3 and 3.5, it is obviously that

Proposition 3.6 Let (\mathbf{H}_1) , (\mathbf{H}_2) be satisfied, $1 - H < \alpha < \alpha_0$ and K is the unit sphere. Then the following assertions are equivalent:

(I) K is viable for the fractional SDE (2.4).

(II) For all $t \in [0, T]$ and all $x \in K$,

$$\langle x, b(t, x) \rangle = 0, \quad \langle x, \sigma(t, x) \rangle = 0.$$

Proposition 3.7 Let (\mathbf{H}_1) , (\mathbf{H}_2) be satisfied, $1 - H < \alpha < \alpha_0$ and K is the unit ball. Then the following assertions are equivalent:

(I) K is viable for the fractional SDE (2.4).

(II) For all $t \in [0, T]$ and all $|x| = 1$,

$$\langle x, b(t, x) \rangle \leq 0, \quad \langle x, \sigma(t, x) \rangle = 0.$$

Corollary 3.8 Consider the SDE on \mathbb{R} ,

$$X_s = x + \int_t^s b(r, X_r) dr + \int_t^s \sigma(r, X_r) dB_r^H, \quad s \in [t, T].$$

$B^H = \{B_t^H, t \geq 0\}$ is a fractional Brownian motion. b, σ satisfy the assumptions $(\mathbf{H}_1), (\mathbf{H}_2)$. Then for any $t \in [0, T]$ and every $x \geq 0$ the equation has a positive solution if and only if

$$b(t, 0) \geq 0, \quad \sigma(t, 0) = 0, \quad \forall t \in [0, T].$$

Proof. In fact we take $K = [0, +\infty)$, the problem is just that K is viable for the fractional SDE. We can use $x = \tan \frac{\pi}{4}(y+1)$ and we get $y = \frac{4}{\pi} \arctan x - 1$, it just maps $[0, +\infty)$ to $[-1, 1]$, and using Proposition 3.7 and Itô formula of fractional SDE (see [13]), we have

$$b(t, 0) \geq 0, \quad \sigma(t, 0) = 0, \quad \forall t \in [0, T].$$

□

The most interesting application is the characterization of comparison theorem. Let us firstly consider the linear case.

Corollary 3.9 *Consider the linear two dimensional decoupled system*

$$\begin{cases} X_s^{t,x} = x + \int_t^s (f(r)X_r^{t,x} + f_1(r))dr + \int_t^s (g(r)X_r^{t,x} + g_1(r))dB_r^H, & s \in [t, T] \\ Y_s^{t,y} = y + \int_t^s (f(r)Y_r^{t,y} + f_2(r))dr + \int_t^s (g(r)Y_r^{t,y} + g_2(r))dB_r^H, & s \in [t, T] \end{cases}$$

then

$$\text{for any } t \in [0, T] \text{ and every } x \leq y, \quad X_s^{t,x} \leq Y_s^{t,y}, \quad \forall s \in [t, T].$$

$$\iff f_1(t) \leq f_2(t), \quad g_1(t) = g_2(t), \quad \forall t \in [0, T].$$

Proof. In fact we set $Z_s^{t,z} = Y_s^{t,y} - X_s^{t,x}$, where $z = y - x \geq 0$, then we can change the problem to $Z_s^{t,z} \geq 0$, it means that for any $t \in [0, T]$ and every $z \geq 0$ the fractional SDE of $Z_s^{t,z}$ has a positive solution. Then using Corollary 3.8, we can easily prove this corollary. \square

In general, we have

Theorem 3.10 (*Comparison theorem*) *Consider the two dimensional decoupled system*

$$\begin{cases} X_s^{t,x} = x + \int_t^s (b_1(r, X_r^{t,x}))dr + \int_t^s (\sigma_1(r, X_r^{t,x}))dB_r^H, & s \in [t, T] \\ Y_s^{t,y} = y + \int_t^s (b_2(r, Y_r^{t,y}))dr + \int_t^s (\sigma_2(r, Y_r^{t,y}))dB_r^H, & s \in [t, T] \end{cases}$$

then

$$\text{for any } t \in [0, T] \text{ and every } x \leq y, \quad X_s^{t,x} \leq Y_s^{t,y}, \quad \forall s \in [t, T]$$

$$\iff b_1(t, z) \leq b_2(t, z), \quad \sigma_1(t, z) = \sigma_2(t, z), \quad \forall t \in [0, T], \quad \forall z \in \mathbb{R}.$$

we will give the proof of this result in Section 5.

4 Proofs of main results

This section is devoted to the proofs of the main results which have been given in Section 3.

Firstly we present some auxiliary Lemmas which will be used in the sequel.

4.1 Auxiliary Results

Lemma 4.1 *Given two stochastic process*

$$U = U^{t,x} : \quad \Omega \times [t, t + \bar{h}] \rightarrow \mathbb{R}^d, \quad U(t) = 0$$

$$V = V^{t,x} : \quad \Omega \times [t, t + \bar{h}] \rightarrow \mathbb{R}^d, \quad V(t) = 0$$

such that for all $s, \tau \in [t, t + \bar{h}]$ and for every $R > 0$ with $|x| \leq R$:

$$|U(s) - U(\tau)| \leq D_R |s - \tau|^{1-\alpha},$$

$$|V(s) - V(\tau)| \leq \tilde{D}_R |s - \tau|^{\min\{\beta, 1-\alpha\}}.$$

then for all $t \leq \tau \leq s \leq t + \bar{h}$,

$$\begin{aligned} (a) \quad & \left| \int_{\tau}^s U(r) dr \right| \leq D_R (s - t)^{1-\alpha} (s - \tau) \\ (b) \quad & \left| \int_{\tau}^s V(r) dB_r^H \right| \leq C_R(\alpha, \beta) \tilde{D}_R \Lambda_{\alpha}(B^H) (s - t)^{\min\{\beta, 1-\alpha\}} (s - \tau)^{1-\alpha}. \end{aligned}$$

where $C_R(\alpha, \beta)$ depends only on R , α , and β .

Proof.

(a) we have

$$\begin{aligned} \left| \int_{\tau}^s U(r) dr \right| &= \left| \int_{\tau}^s [U(r) - U(t)] dr \right| \\ &\leq D_R \left| \int_{\tau}^s (r - t)^{1-\alpha} dr \right| \\ &\leq D_R (s - t)^{1-\alpha} (s - \tau). \end{aligned}$$

(d)

$$\begin{aligned} \left| \int_{\tau}^s V(r) dB_r^H \right| &= \left| \int_{\tau}^s [V(r) - V(t)] dB_r^H \right| \\ &\leq \Lambda_{\alpha}(B^H) \|V\|_{\alpha, 1; [\tau, s]} \\ &\leq \Lambda_{\alpha}(B^H) \int_{\tau}^s \left[\frac{|V(r) - V(t)|}{(r - \tau)^{\alpha}} + \int_{\tau}^r \frac{|V(r) - V(y)|}{(r - y)^{\alpha+1}} dy \right] dr \\ &\leq \tilde{D}_R \Lambda_{\alpha}(B^H) \int_{\tau}^s \left[\frac{(r - t)^{\min\{\beta, 1-\alpha\}}}{(r - \tau)^{\alpha}} + \int_{\tau}^r \frac{(r - y)^{\min\{\beta, 1-\alpha\}}}{(r - y)^{\alpha+1}} dy \right] dr \\ &\leq \tilde{D}_R \Lambda_{\alpha}(B^H) \left[\frac{1}{1 - \alpha} (s - t)^{\min\{\beta, 1-\alpha\}} (s - \tau)^{1-\alpha} \right. \\ &\quad \left. + \int_{\tau}^s \int_{\tau}^r (r - y)^{\min\{\beta - \alpha, 1 - 2\alpha\} - 1} dy dr \right] \\ &\leq C_R(\alpha, \beta) \tilde{D}_R \Lambda_{\alpha}(B^H) (s - t)^{\min\{\beta, 1-\alpha\}} (s - \tau)^{1-\alpha}. \end{aligned}$$

□

Remark. From (a) and (b), just taking $\tau = t$, we have

$$\begin{aligned} (a') \quad & \left| \int_t^s U(r) dr \right| \leq D_R (s - t)^{2-\alpha} \\ (b') \quad & \left| \int_t^s V(r) dB_r^H \right| \leq C_R(\alpha, \beta) \tilde{D}_R \Lambda_{\alpha}(B^H) (s - t)^{1 + \min\{\beta - \alpha, 1 - 2\alpha\}}. \end{aligned}$$

Lemma 4.2 Given two stochastic process $U = U^{t,x}$, $V = V^{t,x}$ which satisfy the conditions in Lemma 4.1, and $\varphi \in \mathcal{H}$, let

$$\begin{aligned} f(r, y) &= \varphi'(y)^+ [U(r) - (\varphi'(y) - \varphi'(x))b(t, x)] \\ g(r, y) &= \varphi'(y)^+ [V(r) - (\varphi'(y) - \varphi'(x))\sigma(t, x)] \end{aligned}$$

then for $\alpha \in (1 - H, \alpha_0)$ and for every $\delta_0 > 0$ there exists a random variable $\bar{h}_1 = \bar{h}_1^{t,x}$ such that for $a_{\varphi} + 2\delta_0 \leq |x| \leq b_{\varphi} - 2\delta_0$ and \mathbb{P} -a.s. $\omega \in \Omega$, the following SDE

$$\xi_s = x + \int_t^s (b(t, x) + f(r, \xi_r)) dr + \int_t^s (\sigma(t, x) + g(r, \xi_r)) dB^H(r), \quad s \in [t, t + \bar{h}_1],$$

has a unique solution $\xi.(\omega) \in L^0(\Omega, \mathcal{F}, \mathbb{P}; W^{\alpha, \infty}(t, T; \mathbb{R}^d))$.
Moreover \mathbb{P} -a.s. $\xi.(\omega) \in C^{1-\alpha}(t, t + \bar{h}_1; \mathbb{R}^d)$.

Proof. From [15] Theorem(the partition of unity) p.61, we have that for every $\delta_0 > 0$, there exists one function $\alpha(x) \in C^\infty(\mathbb{R}^d)$ such that $\alpha(x) = 1$ for $a_\varphi + \delta_0 \leq |x| \leq b_\varphi - \delta_0$ and $\alpha(x) = 0$ for $|x| \geq b_\varphi$ or $|x| \leq a_\varphi$, then we define

$$\tilde{f}(t, y) = \alpha(y)f(t, y) = \begin{cases} f(t, y), & a_\varphi + \delta_0 \leq |y| \leq b_\varphi - \delta_0 \\ \alpha(y)f(t, y) & a_\varphi \leq |y| \leq a_\varphi + \delta_0, \text{ or } b_\varphi - \delta_0 \leq |y| \leq b_\varphi \\ 0, & |y| \geq b_\varphi, \text{ or } |y| \leq a_\varphi. \end{cases}$$

and we define $\tilde{g}(t, y)$ in the same method and then we consider the following SDE

$$\tilde{\xi}_s = x + \int_t^s (b(t, x) + \tilde{f}(r, \tilde{\xi}_r))dr + \int_t^s (\sigma(t, x) + \tilde{g}(r, \tilde{\xi}_r))dB^H(r), \quad s \in [t, t + \bar{h}]. \quad (4.1)$$

Since $\varphi \in \mathcal{H}$ and $U = U^{t, x}, V = V^{t, x}$ satisfy the conditions in Lemma 4.1, we can verify that for $\alpha \in (1 - H, \alpha_0)$, $\tilde{f}(t, y)$, and $\tilde{g}(t, y)$ satisfy the conditions in $(\mathbf{H}_1), (\mathbf{H}_2)$ in [10] where the constants M_0, M_R, L_0, L_R depend on ω , then the SDE (4.1) has a unique solution $\tilde{\xi}.(\omega) \in L^0(\Omega, \mathcal{F}, \mathbb{P}; W^{\alpha, \infty}(t, T; \mathbb{R}^d))$ for all $\alpha \in (1 - H, \alpha_0)$. And moreover \mathbb{P} -a.s. $\tilde{\xi}.(\omega) \in C^{1-\alpha}(t, t + \bar{h}; \mathbb{R}^d)$. Since $a_\varphi + 2\delta_0 \leq |x| \leq b_\varphi - 2\delta_0$ then there exists a random variable $\bar{h}_1 = \bar{h}_1^{t, x}$, such that \mathbb{P} -a.s. $a_\varphi + \delta_0 \leq |\tilde{\xi}| \leq b_\varphi - \delta_0$, then for $s \in [t, t + \bar{h}_1]$, the SDE (4.1) becomes \mathbb{P} -a.s.

$$\tilde{\xi}_s = x + \int_t^s (b(t, x) + f(r, \tilde{\xi}_r))dr + \int_t^s (\sigma(t, x) + g(r, \tilde{\xi}_r))dB^H(r), \quad s \in [t, t + \bar{h}_1].$$

just taking $\xi_s = \tilde{\xi}_s, s \in [t, t + \bar{h}_1]$, and together with the uniqueness of $\tilde{\xi}_s$, then we finish our proof. □

4.2 Proof of Theorem 3.1 and Theorem 3.2

Proof of Theorem 3.1

Since $(b(t, x), \sigma(t, x)) \in S_{K(t)}(t, x)$, then there exist a random variable $\bar{h} = \bar{h}^{t, x} > 0$, and two stochastic process

$$U = U^{t, x} : \Omega \times [t, t + \bar{h}] \rightarrow \mathbb{R}^d, \quad U(t) = 0$$

$$V = V^{t, x} : \Omega \times [t, t + \bar{h}] \rightarrow \mathbb{R}^d, \quad V(t) = 0$$

such that for all $s, \tau \in [t, t + \bar{h}]$ and for every $R > 0$ and $|x| \leq R$:

$$|U(s) - U(\tau)| \leq D_R |s - \tau|^{1-\alpha}, \quad |V(s) - V(\tau)| \leq \tilde{D}_R |s - \tau|^{\min\{\beta, 1-\alpha\}}$$

and

$$x + \int_t^s (b(t, x) + U(r))dr + \int_t^s (\sigma(t, x) + V(r))dB^H(r) \in K(s),$$

where D_R, \tilde{D}_R , depend only on $R, L_R, M_{0, T}, M_0, L_0, T, \alpha, \beta, \Lambda_\alpha(B^H)$.

Let

$$\eta_s = x + \int_t^s (b(t, x) + U(r))dr + \int_t^s (\sigma(t, x) + V(r))dB^H(r)$$

and from Lemma 4.1 and $H - \epsilon$ Hölder continuous property of fractional Brownian motion, it follows that for all $s, \tau \in [t, t + \bar{h}]$,

$$|\eta_s - \eta_\tau| \leq \zeta(s - \tau)^{1-\alpha}.$$

According to the fractional Itô formula (see Yuliya S.Mishura [13]), We have for all $s \in [t, t + \bar{h}]$

$$\begin{aligned} & \varphi\left(x + \int_t^s (b(t, x) + U(r))dr + \int_t^s (\sigma(t, x) + V(r))dB^H(r)\right) \\ &= \varphi(x) + \int_t^s [\varphi'(\eta_r)(b(t, x) + U(r))]dr + \int_t^s [\varphi'(\eta_r)(\sigma(t, x) + V(r))]dB^H(r) \\ &= \varphi(x) + \int_t^s [\varphi'(x)b(t, x) + U_1(r)]dr + \int_t^s [\varphi'(x)\sigma(t, x) + V_1(r)]dB^H(r) \end{aligned}$$

where

$$\begin{aligned} U_1(r) &= \varphi'(\eta_r)U(r) + (\varphi'(\eta_r) - \varphi'(x))b(t, x) \\ V_1(r) &= \varphi'(\eta_r)V(r) + (\varphi'(\eta_r) - \varphi'(x))\sigma(t, x) \end{aligned}$$

Then

$$\begin{aligned} & \varphi(x) + \int_t^s [\varphi'(x)b(t, x) + U_1(r)]dr + \int_t^s [\varphi'(x)\sigma(t, x) + V_1(r)]dB^H(r) \\ &= \varphi\left(x + \int_t^s (b(t, x) + U(r))dr + \int_t^s (\sigma(t, x) + V(r))dB^H(r)\right) \in \varphi(K(s)) \end{aligned}$$

and it's easy to verify that

$$U_1(t) = 0, \quad V_1(t) = 0$$

For all $s, \tau \in [t, t + \bar{h}]$ and for every $R > 0$ and $|x| \leq R$, Using the Lipschitz continuity of φ' and (\mathbf{H}_2) , we obtain that

$$\begin{aligned} |U_1(s) - U_1(\tau)| &\leq |\varphi'(\eta_\tau)||U(s) - U(\tau)| + (|U(s)| + |b(t, x)|)|\varphi'(\eta_s) - \varphi'(\eta_\tau)| \\ &\leq \theta_1 |s - \tau|^{1-\alpha} + \theta_2 |\eta_s - \eta_\tau| \\ &\leq \theta |s - \tau|^{1-\alpha} \end{aligned}$$

Similarly we can proof that

$$|V_1(s) - V_1(\tau)| \leq \tilde{\theta} |s - \tau|^{\min\{\beta, 1-\alpha\}}$$

The Hölder constants $\theta, \tilde{\theta}$ are random variables which depend only on $R, L_R, M_0, M_0, L_0, T, \alpha, \beta, \Lambda_\alpha(B^H)$.

This means that

$$(\varphi'(x)b(t, x), \varphi'(x)\sigma(t, x)) \in S_{\varphi(K(t))}(t, \varphi(x)).$$

□

Proof of Theorem 3.2

We shall only have to prove that from $(\varphi'(x)b(t, x), \varphi'(x)\sigma(t, x)) \in S_{\varphi(K(t))}(t, \varphi(x))$, we infer that $(b(t, x), \sigma(t, x)) \in S_{\varphi^{-1}(\varphi(K(t)))}(t, x)$.

Since

$$(\varphi'(x)b(t, x), \varphi'(x)\sigma(t, x)) \in S_{\varphi(K(t))}(t, \varphi(x)).$$

then for $x \in K(t)$, there exist a random variable $\bar{h} = \bar{h}^{t,x} > 0$ and two stochastic process,

$$U_1 = U_1^{t,x} : \Omega \times [t, t + \bar{h}] \rightarrow \mathbb{R}^m, \quad U_1(t) = 0,$$

$$V_1 = V_1^{t,x} : \Omega \times [t, t + \bar{h}] \rightarrow \mathbb{R}^m, \quad V_1(t) = 0$$

such that for all $s, \tau \in [t, t + \bar{h}]$ and for every $R > 0$ and $|x| \leq R$,

$$|U_1(s) - U_1(\tau)| \leq D_R |s - \tau|^{1-\alpha}, \quad |V_1(s) - V_1(\tau)| \leq \tilde{D}_R |s - \tau|^{\min\{\beta, 1-\alpha\}}$$

and

$$\varphi(x) + \int_t^s (\varphi'(x)b(t, x) + U_1(r))dr + \int_t^s (\varphi'(x)\sigma(t, x) + V_1(r))dB^H(r) \in \varphi(K(s)).$$

Let

$$\begin{aligned} f(r, y) &= \varphi'(y)^+ [U_1(r) - (\varphi'(y) - \varphi'(x))b(t, x)], \\ g(r, y) &= \varphi'(y)^+ [V_1(r) - (\varphi'(y) - \varphi'(x))\sigma(t, x)], \end{aligned}$$

where $\varphi'(y)^+$ is the right inverse of $\varphi'(y)$. By Lemma 4.2, for every $\delta_0 > 0$ and $a_\varphi + 2\delta_0 \leq |x| \leq b_\varphi - 2\delta_0$, there exists a random variable \bar{h}_1 such that for \mathbb{P} -a.s. $\omega \in \Omega$ the following SDE

$$\xi_s = x + \int_t^s (b(t, x) + f(r, \xi_r))dr + \int_t^s (\sigma(t, x) + g(r, \xi_r))dB^H(r), \quad s \in [t, t + \bar{h}_1],$$

has a unique solution $\xi.(\omega)$. Then with

$$\begin{aligned} U(r) &= \varphi'(\xi_r)^+ [U_1(r) - (\varphi'(\xi_r) - \varphi'(x))b(t, x)] \quad \text{and} \\ V(r) &= \varphi'(\xi_r)^+ [V_1(r) - (\varphi'(\xi_r) - \varphi'(x))\sigma(t, x)] \end{aligned}$$

according to the fractional Itô formula, we have for all $s \in [t, t + \bar{h}_1]$

$$\begin{aligned} &\varphi\left(x + \int_t^s (b(t, x) + U(r))dr + \int_t^s (\sigma(t, x) + V(r))dB^H(r)\right) \\ &= \varphi(x) + \int_t^s (\varphi'(x)b(t, x) + U_1(r))dr + \int_t^s (\varphi'(x)\sigma(t, x) + V_1(r))dB^H(r) \in \varphi(K(s)). \end{aligned}$$

Clearly that

$$U(t) = 0, \quad V(t) = 0.$$

Since $\varphi \in \mathcal{H}$ and $\xi.(\omega) \in C^{1-\alpha}(t, t + \bar{h}_1; \mathbb{R}^d)$ and together with (\mathbf{H}_1) and (\mathbf{H}_2) , it easily follows

$$|U(s) - U(\tau)| \leq \theta |s - \tau|^{1-\alpha}, \quad |V(s) - V(\tau)| \leq \tilde{\theta} |s - \tau|^{\min\{\beta, 1-\alpha\}}.$$

Then it means that

$$(b(t, x), \sigma(t, x)) \in S_{\varphi^{-1}(\varphi(K(t)))}(t, x).$$

□

4.3 Proof of Lemmas 3.3, 3.4, 3.5

. Proof of Lemma 3.3

Firstly, we take $\varphi(x) = |x|^2$, and it's easy to verify that for $\frac{1}{4} \leq |x| \leq 4$,

$$\varphi'(x)^+ = \frac{x}{2|x|^2}.$$

and we can verify that $\varphi \in \mathcal{H}$ taking $a_\varphi = \frac{1}{4}$, $b_\varphi = 4$, $\varepsilon = \frac{1}{4}$, then for $x \in K$ we have $\frac{1}{2} \leq |x| = 1 \leq 4 - \frac{1}{4}$, by Theorem 3.2 we have

$$(b(t, x), \sigma(t, x)) \in S_K(t, x) \Leftrightarrow (\langle 2x, b(t, x) \rangle, 2x^* \sigma(t, x)) \in S_1(t, x^2)$$

So now it's equivalent to prove

$$(\langle 2x, b(t, x) \rangle, \langle 2x, \sigma(t, x) \rangle) \in S_1(t, |x|^2) \Leftrightarrow \langle x, b(t, x) \rangle = 0, \quad \langle x, \sigma(t, x) \rangle = 0$$

Sufficient. If $\langle x, b(t, x) \rangle = 0$, $\langle x, \sigma(t, x) \rangle = 0$, we can take $U(r) \equiv 0$, $V(r) \equiv 0$, and we have $\forall s \in [t, t + \bar{h}]$ and $|x| = 1$

$$|x|^2 + \int_t^s (\langle 2x, b(t, x) \rangle + U(r))dr + \int_t^s (\langle 2x, \sigma(t, x) \rangle + V(r))dB^H(r) = 1,$$

This means that $(\langle 2x, b(t, x) \rangle, \langle 2x, \sigma(t, x) \rangle) \in S_1(t, |x|^2)$.

Necessary. Since $(\langle 2x, b(t, x) \rangle, \langle 2x, \sigma(t, x) \rangle) \in S_1(t, |x|^2)$. then there exist a random variable $\bar{h} = \bar{h}^{t, x} > 0$, and two stochastic process

$$U = U^{t, x} : \Omega \times [t, t + \bar{h}] \rightarrow \mathbb{R}, \quad U(t) = 0$$

$$V = V^{t, x} : \Omega \times [t, t + \bar{h}] \rightarrow \mathbb{R}, \quad V(t) = 0$$

such that for all $s, \tau \in [t, t + \bar{h}]$ and for every $R > 0$, $|x| \leq R$:

$$|U(s) - U(\tau)| \leq D_R |s - \tau|^{1-\alpha}, \quad |V(s) - V(\tau)| \leq \tilde{D}_R |s - \tau|^{\min\{\beta, 1-\alpha\}}$$

and

$$|x|^2 + \int_t^s (\langle 2x, b(t, x) \rangle + U(r))dr + \int_t^s (\langle 2x, \sigma(t, x) \rangle + V(r))dB^H(r) = 1, \quad \mathbb{P} - a.s. \quad (4.2)$$

where D_R, \tilde{D}_R , depend only on $R, L_R, M_0, L_0, T, \alpha, \beta, \Lambda_\alpha(B^H)$.

Since $|x|^2 = 1$, then from the equation (4.2) we clearly have

$$\left| \langle 2x, b(t, x) \rangle + \left[\int_t^s U(r)dr + \int_t^s V(r)dB^H(r) \right] \frac{1}{s-t} \right| = \left| \langle 2x, \sigma(t, x) \rangle \frac{B^H(s) - B^H(t)}{s-t} \right| \quad (4.3)$$

By (2.2), there exists $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that $\forall \omega \in \Omega_0$, (4.2) is satisfied and

$$\limsup_{t \rightarrow t_0, t \geq t_0} \left| \frac{B_t^H(\omega) - B_{t_0}^H(\omega)}{t - t_0} \right| = +\infty.$$

Let $\omega_0 \in \Omega_0$. Then there is a subsequence $r_n = r_n(\omega_0) \downarrow t$ when $n \rightarrow \infty$, such that

$$\lim_{n \rightarrow \infty} \left| \frac{B_{r_n}^H(\omega_0) - B_t^H(\omega_0)}{r_n - t} \right| = +\infty. \quad (4.4)$$

Setting in 4.3 $s = r_n \wedge (t + \bar{h}(\omega_0)) \in [t, t + \bar{h}(\omega_0)]$ and passing to limit as $n \rightarrow \infty$, the left member, via Lemma 4.1, has limit $2\langle x, b(t, x) \rangle$. Consequently, noting (4.4), we must have

$$\langle x, \sigma(t, x) \rangle = 0.$$

and therefore

$$\langle x, b(t, x) \rangle = 0.$$

The proof of Lemma 3.3 is complete. □

Proof of Lemma 3.4

Like the analysis in the proof of Lemma 3.3, the proof of Lemma 3.4 is reduced to the following equivalent:

$\forall x$ such that $|x| = R$

$$(\langle 2x, b(t, x) \rangle, 2x^* \sigma(t, x)) \in S_{\varphi(K)}(t, |x|^2) \Leftrightarrow \langle x, b(t, x) \rangle \leq 0, \quad \langle x, \sigma(t, x) \rangle = 0$$

$\forall x$, such that $|x| = r$,

$$(\langle 2x, b(t, x) \rangle, 2x^* \sigma(t, x)) \in S_{\varphi(K)}(t, |x|^2) \Leftrightarrow \langle x, b(t, x) \rangle \geq 0, \quad \langle x, \sigma(t, x) \rangle = 0.$$

We only prove in the case $|x| = R$, the other one is similar.

Sufficient. If $\langle x, b(t, x) \rangle \leq 0$, $\langle x, \sigma(t, x) \rangle = 0$, taking $U(r) \equiv 0$, $V(r) \equiv 0$, and we can choose \bar{h} small enough such that $\forall s \in [t, t + \bar{h}]$,

$$r^2 \leq |x|^2 + \int_t^s (\langle 2x, b(t, x) \rangle + U(r))dr + \int_t^s (2x^* \sigma(t, x) + V(r))dB^H(r) \leq R^2,$$

This means that $(\langle 2x, b(t, x) \rangle, \langle 2x, \sigma(t, x) \rangle) \in S_{\varphi(K)}(t, |x|^2)$.

Necessary. Since $(\langle 2x, b(t, x) \rangle, \langle 2x, \sigma(t, x) \rangle) \in S_{\varphi(K)}(t, |x|^2)$. Then there exist random variable $\bar{h} = \bar{h}^{t,x} > 0$, and two stochastic process

$$U = U^{t,x} : \Omega \times [t, t + \bar{h}] \rightarrow \mathbb{R}, \quad U(t) = 0$$

$$V = V^{t,x} : \Omega \times [t, t + \bar{h}] \rightarrow \mathbb{R}, \quad V(t) = 0$$

such that for all $s, \tau \in [t, t + \bar{h}]$ and for every $R > 0$, $|x| \leq R$:

$$|U(s) - U(\tau)| \leq D_R |s - \tau|^{1-\alpha}, \quad |V(s) - V(\tau)| \leq \tilde{D}_R |s - \tau|^{\min\{\beta, 1-\alpha\}}$$

and

$$r^2 \leq |x|^2 + \int_t^s (\langle 2x, b(t, x) \rangle + U(r))dr + \int_t^s (\langle 2x, \sigma(t, x) \rangle + V(r))dB^H(r) \leq R^2,$$

where D_R, \tilde{D}_R , depend only on $R, L_R, M_0, L_0, T, \alpha, \beta, \Lambda_\alpha(B^H)$.

Since $|x| = R$, then we yield

$$\langle 2x, b(t, x) \rangle(s - t) + \langle 2x, \sigma(t, x) \rangle(B^H(s) - B^H(t)) + \int_t^s U(r)dr + \int_t^s V(r)dB^H(r) \leq 0 \quad (4.5)$$

By (2.3), there exists $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = \frac{1}{2}$ such that for each $\omega_0 \in \Omega_0$, (4.5) is satisfied and there is a sequence $t \leq s_n = s_n(\omega_0) \leq t + \bar{h}(\omega_0)$, $s_n \downarrow t$, such that

$$\lim_{s_n \downarrow t} \frac{B_{s_n}^H(\omega_0) - B_t^H(\omega_0)}{s_n - t} = +\infty. \quad (4.6)$$

Then we have

$$\langle 2x, \sigma(t, x) \rangle \frac{B_{\omega_0}^H(s_n) - B_{\omega_0}^H(t)}{s_n - t} + \left[\int_t^{s_n} U(r) dr + \int_t^{s_n} V(r) dB_{\omega_0}^H(r) \right] \frac{1}{s_n - t} \leq -\langle 2x, b(t, x) \rangle. \quad (4.7)$$

By Lemma 4.1

$$\left[\int_t^{s_n} U(r) dr + \int_t^{s_n} V(r) dB_{\omega_0}^H(r) \right] \frac{1}{s_n - t} \rightarrow 0.$$

and noting (4.6), we derive that

$$\langle x, \sigma(t, x) \rangle \leq 0.$$

Similarly we can prove $\langle x, \sigma(t, x) \rangle \geq 0$, choosing ω'_0 and a sequence $t \leq r_n = r_n(\omega'_0) \leq t + \bar{h}(\omega'_0)$ and $r_n \downarrow t$ such that $\lim_{r_n \downarrow t} \frac{B_{r_n}^H(\omega'_0) - B_t^H(\omega'_0)}{r_n - t} = -\infty$. So

$$\langle x, \sigma(t, x) \rangle = 0.$$

Then from (4.5), we have

$$\langle 2x, b(t, x) \rangle + \left[\int_t^s U(r) dr + \int_t^s V(r) dB^H(r) \right] \frac{1}{s - t} \leq 0.$$

and passing to limit $s \rightarrow t$, it follows, via Lemma 4.1,

$$\langle x, b(t, x) \rangle \leq 0$$

The proof of Lemma 3.4 is finished. □

Proof of Lemma 3.5 It is very similar to the proof of Lemma 3.4, therefore we omit it. □

5 Proof of the Comparison Theorem

Proof of Theorem 3.10

We write the two dimensional decoupled system

$$\begin{cases} X_s^{t,x} = x + \int_t^s (b_1(r, X_r^{t,x})) dr + \int_t^s (\sigma_1(r, X_r^{t,x})) dB_r^H, & s \in [t, T] \\ Y_s^{t,y} = y + \int_t^s (b_2(r, Y_r^{t,y})) dr + \int_t^s (\sigma_2(r, Y_r^{t,y})) dB_r^H, & s \in [t, T] \end{cases}$$

as

$$Z_s^{t,z} = z + \int_t^s (b(r, Z_r^{t,z})) dr + \int_t^s (\sigma(r, Z_r^{t,z})) dB_r^H, \quad s \in [t, T]$$

where

$$Z_s^{t,z} = \begin{pmatrix} X_s^{t,x} \\ Y_s^{t,y} \end{pmatrix}, \quad z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad b(r, Z_r^{t,z}) = \begin{pmatrix} b_1(r, X_r^{t,x}) \\ b_2(r, Y_r^{t,y}) \end{pmatrix}, \quad \sigma(r, Z_r^{t,z}) = \begin{pmatrix} \sigma_1(r, X_r^{t,x}) \\ \sigma_2(r, Y_r^{t,y}) \end{pmatrix}.$$

we take $\varphi(z) = \varphi(x, y) = y - x$, then for every $z \in \mathbb{R}^2$, $\varphi'(z) = (-1, 1)$, and

$$\varphi'(z)^+ = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

so $\varphi \in \mathcal{H}$ and if we set $K = \{(x, y) \mid y - x \geq 0\}$, we have $\varphi(K) = \mathbb{R}^+$ and $\varphi^{-1}(\varphi(K)) = K$, then using the same method of the proof of Theorem 3.2 we have

$$(b(t, z), \sigma(t, z)) \in S_K(t, x) \Leftrightarrow (\varphi'(z)b(t, z), \varphi'(z)\sigma(t, z)) \in S_{\mathbb{R}^+}(t, \varphi(z)).$$

In fact, considering Theorem 3.1, we only need prove

$$(\varphi'(z)b(t, z), \varphi'(z)\sigma(t, z)) \in S_{\mathbb{R}^+}(t, \varphi(z)) \Rightarrow (b(t, z), \sigma(t, z)) \in S_K(t, x).$$

Since

$$(\varphi'(z)b(t, z), \varphi'(z)\sigma(t, z)) \in S_{\mathbb{R}^+}(t, \varphi(z))$$

then for $z \in K$, there exist a random variable $\bar{h} = \bar{h}^{t, z} > 0$, and two stochastic process,

$$U_1 = U_1^{t, z} : \Omega \times [t, t + \bar{h}] \rightarrow \mathbb{R}, \quad U_1(t) = 0$$

$$V_1 = V_1^{t, z} : \Omega \times [t, t + \bar{h}] \rightarrow \mathbb{R}, \quad V_1(t) = 0$$

such that for all $s, \tau \in [t, t + \bar{h}]$ and for every $R > 0$ and $|z| \leq R$,

$$|U_1(s) - U_1(\tau)| \leq D_R |s - \tau|^{1-\alpha}, \quad |V_1(s) - V_1(\tau)| \leq \tilde{D}_R |s - \tau|^{\min\{\beta, 1-\alpha\}}$$

and

$$\varphi(z) + \int_t^s (\varphi'(z)b(t, z) + U_1(r))dr + \int_t^s (\varphi'(z)\sigma(t, z) + V_1(r))dB^H(r) \in \varphi(K(s)).$$

Let

$$\begin{aligned} f(r, y) &= \varphi'(y)^+ [U_1(r) - (\varphi'(y) - \varphi'(z))b(t, z)] \\ g(r, y) &= \varphi'(y)^+ [V_1(r) - (\varphi'(y) - \varphi'(z))\sigma(t, z)]. \end{aligned}$$

It's obviously that $f(r, y), g(r, y)$ are independent of y . Let

$$\xi_s = z + \int_t^s (b(t, z) + f(r, \xi_r))dr + \int_t^s (\sigma(t, z) + g(r, \xi_r))dB^H(r), \quad s \in [t, t + \bar{h}], \quad |z| \leq R.$$

Then we take

$$\begin{aligned} U(r) &= \varphi'(\xi_r)^+ [U_1(r) - (\varphi'(\xi_r) - \varphi'(z))b(t, z)] \\ V(r) &= \varphi'(\xi_r)^+ [V_1(r) - (\varphi'(\xi_r) - \varphi'(z))\sigma(t, z)] \end{aligned}$$

According to the fractional Itô formula, we have for all $s \in [t, t + \bar{h}]$

$$\begin{aligned} &\varphi\left(z + \int_t^s (b(t, z) + U(r))dr + \int_t^s (\sigma(t, z) + V(r))dB^H(r)\right) \\ &= \varphi(z) + \int_t^s (\varphi'(z)b(t, z) + U_1(r))dr + \int_t^s (\varphi'(z)\sigma(t, z) + V_1(r))dB^H(r) \in \mathbb{R}^+ \end{aligned}$$

Clearly that

$$U(t) = 0, \quad V(t) = 0.$$

Since for every $z \in \mathbb{R}^2$, $\varphi'(z) = (-1, 1)$, $\varphi'(z)^+ = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and together with $(\mathbf{H}_1), (\mathbf{H}_2)$, it is clear that

$$|U(s) - U(\tau)| \leq \theta |s - \tau|^{1-\alpha}, \quad |V(s) - V(\tau)| \leq \tilde{\theta} |s - \tau|^{\min\{\beta, 1-\alpha\}}.$$

This means that

$$(b(t, z), \sigma(t, z)) \in S_{\varphi^{-1}(\varphi(K))}(t, x) = S_K(t, x).$$

Just as Corollary 2.7 said, if we want to get the conditions for the viability of K , we only need to think about the starting point $x \in \partial K$. Then the comparison theorem is equivalent to prove that for all $t \in [0, T]$ and for any $z = \begin{pmatrix} x \\ y \end{pmatrix}$ such that $x = y$, and $|z| \leq R$,

$$(\varphi'(z)b(t, z), \varphi'(z)\sigma(t, z)) \in S_{\mathbb{R}^+}(t, \varphi(z)) \Leftrightarrow b_1(t, x) \leq b_2(t, y), \sigma_1(t, x) = \sigma_2(t, y).$$

Sufficient. If $b_1(t, x) \leq b_2(t, y)$, $\sigma_1(t, x) = \sigma_2(t, y)$, for $x = y$, we can take $U(r) \equiv 0$, $V(r) \equiv 0$, and we have $\forall s \in [t, t + \bar{h}]$, and $z = \begin{pmatrix} x \\ y \end{pmatrix}$, such that $x = y$,

$$y - x + \int_t^s (\varphi'(z)b(t, z) + U(r))dr + \int_t^s (\varphi'(z)\sigma(t, z) + V(r))dB^H(r) = (b_2(t, y) - b_1(t, y))(s - t) \geq 0,$$

This means that $(\varphi'(z)b(t, z), \varphi'(z)\sigma(t, z)) \in S_{\mathbb{R}^+}(t, \varphi(z))$.

Necessary. Since $(\varphi'(z)b(t, z), \varphi'(z)\sigma(t, z)) \in S_{\mathbb{R}^+}(t, \varphi(z))$, then there exist random variable $\bar{h} = \bar{h}^{t, z} > 0$, and two stochastic process

$$U = U^{t, z} : \Omega \times [t, t + \bar{h}] \rightarrow \mathbb{R}, \quad U(t) = 0$$

$$V = V^{t, z} : \Omega \times [t, t + \bar{h}] \rightarrow \mathbb{R}, \quad V(t) = 0$$

such that for all $s, \tau \in [t, t + \bar{h}]$ and for every $R > 0$ and $|z| \leq R$:

$$|U(s) - U(\tau)| \leq D_R |s - \tau|^{1-\alpha}, \quad |V(s) - V(\tau)| \leq \tilde{D}_R |s - \tau|^{\min\{\beta, 1-\alpha\}}$$

and

$$y - x + \int_t^s ((b_2(t, y) - b_1(t, x)) + U(r))dr + \int_t^s ((\sigma_2(t, y) - \sigma_1(t, x)) + V(r))dB^H(r) \geq 0,$$

Since $y = x$, then we get

$$(b_2(t, x) - b_1(t, x))(s - t) + (\sigma_2(t, x) - \sigma_1(t, x))(B^H(s) - B^H(t)) + \int_t^s U(r)dr + \int_t^s V(r)dB^H(r) \geq 0.$$

With the same analysis in the proof of Lemma 3.4, we obtain that for every $R > 0$

$$b_1(t, x) \leq b_2(t, x), \quad \sigma_1(t, x) = \sigma_2(t, x), \quad \forall |x| \leq R.$$

This complete the proof of Comparison Theorem. □

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